EE 3054 Lab 4: Complex Poles

Before you begin this lab, please read the notes on *Partial Fraction Expansion.* These notes show how to use the Matlab `residue` command.

Given the transfer function $H(z)$ of a causal discrete-time LTI system, how do you find the impulse response $h(n)$? In this lab, we cover three ways get the impulse response from the transfer function. For the following problems, please use the following transfer function,

$$H(z) = \frac{1 - 2.5\, z^{-1} + z^{-2}}{1 - z^{-1} + 0.7\, z^{-2}}.$$

1. What is the difference equation that implements this system? In Lab 3 you wrote a Matlab function that implements a difference equation. Modify that function so that it implements the difference equation for the system considered here. Use your new program to numerically compute the impulse response of this system. Once you have computed the impulse response, make a stem plot of it. By the way, Matlab has a built in command `filter` that implements a difference equation. Use the `help` command to find out how to use the `filter` command. Use the `filter` command to compute the impulse response. Does it agree with the one computed using your own program? (It should!)

2. A second way to get $h(n)$ from $H(z)$ is to use partial fraction expansion. Partial fraction expansion can be performed in Matlab with the `residue` command. Use this command to perform partial fraction expansion on $H(z)/z$. Then you can write the impulse response as

$$h(n) = C_1\, p_1^n\, u(n) + C_2\, p_2^n\, u(n). \qquad (1)$$

   The four values C_1, C_2, p_1, p_2 are found using the `residue` command. For this example they will be complex! Use Equation (1) to compute in Matlab the impulse response,

```
n = -5:30;
h = C1 * p1.^n .* (n>=0) + ...
```

   Note that even though C_1, C_2, p_1, p_2 are complex, the impulse response $h(n)$ is real-valued. (The imaginary parts cancel out.) Is this what you find? Make a stem plot of the impulse response you have computed using Equation (1). Verify that the impulse response is the same as the one you computed in part (a).

3. In this part you are to write a formula for the impulse response that does not involve complex numbers. The impulse response you have found can also be written as

$$h(n) = A\, r^n \cos(\omega_o\, n + \theta_o)\, u(n). \tag{2}$$

This is a damped sinusoid. This formula is obtained from Equation (1) by putting the complex values C_1, C_2, p_1, p_2 into polar form:

$$\begin{aligned} C_1 &= R_1\, e^{j\,\alpha_1} \\ C_2 &= R_2\, e^{j\,\alpha_2} \\ p_1 &= r_1\, e^{j\,\beta_1} \\ p_2 &= r_2\, e^{j\,\beta_2}. \end{aligned}$$

Note about complex numbers: to put a complex number c into polar form ($c = r\, e^{j\,\alpha}$) we set $r = |c|$ and $\alpha = \angle c$. In Matlab, $|c|$ is computed with the command `abs(c)` and $\angle c$ is computed with the command `angle(c)`.

In part (b) you found the complex values C_1, etc. Now find the real values R_1, α_1, etc, by using the commands `abs` and `angle`. You should find that $R_2 = R_1$, $\alpha_2 = -\alpha_1$, $r_2 = r_1$, and $\beta_2 = -\beta_1$. Is this what you find? Therefore, the formula in Equation (1) becomes

$$\begin{aligned} h(n) &= R_1\, e^{j\,\alpha_1}\, (r_1\, e^{j\,\beta_1})^n\, u(n) + R_1\, e^{-j\,\alpha_1}\, (r_1\, e^{-j\,\beta_1})^n\, u(n) \\ &= R_1\, e^{j\,\alpha_1}\, r_1^n\, e^{j\,\beta_1\, n}\, u(n) + R_1\, e^{-j\,\alpha_1}\, r_1^n\, e^{-j\,\beta_1\, n}\, u(n) \\ &= R_1\, r_1^n\, (e^{j(\beta_1\, n+\alpha_1)} + e^{-j(\beta_1\, n+\alpha_1)})\, u(n) \\ &= 2\, R_1\, r_1^n\, \cos(\beta_1\, n + \alpha_1)\, u(n). \end{aligned}$$

This finally has the form in Equation (2). The equation now contains no complex numbers. Although the derivation looks messy, each step uses a basic identity which you should know. The last step uses the complex form for cosine.

In Matlab, find the real constants in Equation (2) and use this equation to compute the impulse response,

```
n = -5:30;
h = A * r.^n .* ...
```

Also, make a stem plot of the impulse response. It should be exactly the same as what you got in parts (a) and (b).

4. Please note the following!

   (a) The frequency ω_o of the damped cosine in Equation (2) is exactly the angle of the pole p_1.

   (b) The decay of the damped cosine depends on r which is exactly the absolute value (modulus) of the pole p_1.

   Therefore, the locations of the poles of an LTI system provides information about the form of the impulse response.

   To make a diagram of the poles and zeros of the system, use the `zplane` command.

   ```
   zplane(b,a)
   ```

   where `b` and `a` are the same vectors used for the `filter` and `residue` commands. (Note: when using `zplane` the vectors `b` and `a` must be *row* vectors. If they are column vectors you must take the transpose of them to get row vectors.)

   Make a diagram of the poles and zeros of the above system.

5. The system considered above was a very simple one. For additional practice, take the transfer function

   $$H(z) = \frac{1 - 0.6\, z^{-1}}{1 - 2.1\, z^{-1} + 1.6\, z^{-2} - 0.4\, z^{-3}}.$$

   For this system repeat the previous parts. For this system Equation (1) will have three components instead of two. Equation (2) will have an extra term:

   $$h(n) = A\, r^n \, \cos(\omega_o\, n + \theta_o)\, u(n) + B\, p_3^n\, u(n).$$

To turn in: the plots you produced, the Matlab code to produce the plots and intermediate results (show your use of the `residue` and `filter` commands). Also include discussion of your observations, explanation of your steps, and derivation of the impulse response in Equation (1).