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6. Find the DTFT $X^f(\omega)$ for the following discrete-time signals.

   (a) $x(n) = \cos(0.2\,\pi\, n + 0.3\,\pi)$

   (b) $x(n) = \cos(0.3\,\pi\, n)\, \sin(0.4\,\pi\, n)$

   Solution:

   (a) Note:

   $$x(n) = \cos(0.2\,\pi\, n + 0.3\,\pi) = 0.5\, e^{j\,0.2\pi n}\, e^{j\,0.3\pi} + 0.5 e^{-j\,0.2\pi n}\, e^{-j\,0.3\pi}$$

   and

   $$g(n) = e^{j\omega_o n} \qquad \longrightarrow \qquad G^f(\omega) = 2\pi\delta(\omega - \omega_o)$$

   so

   $$X^f(\omega) = \pi e^{j\,0.3\pi}\delta(\omega - 0.2\pi) + \pi e^{-j\,0.3\pi}\delta(\omega + 0.2\pi)$$

   (b)

   $$x(n) = \cos(0.3\,\pi\, n)\, \sin(0.4\,\pi\, n) = (0.5\, e^{j\,0.3\pi n} + 0.5 e^{-j\,0.3\pi n})((0.5/j)\, e^{j\,0.4\pi n} - (0.5/j) e^{-j\,0.4\pi n})$$

   $$x(n) = -\frac{j}{4}(e^{j\,0.7\pi n} + e^{j\,0.1\pi n} - e^{-j\,0.1\pi n} - e^{-j\,0.7\pi n})$$

   (note that $1/j = -j$).

   $$X^f(\omega) = j\frac{\pi}{2}(\delta(\omega + 0.7\,\pi) + \delta(\omega + 0.1\,\pi) - \delta(\omega - 0.1\,\pi) - \delta(\omega - 0.7\,\pi))$$

---

7. Given the following DTFT $X^f(\omega)$ over the interval $|\omega| \le \pi$, find the signal $x(n)$.

   (a) $X^f(\omega) = \delta(\omega + \frac{\pi}{4}) + \delta(\omega - \frac{\pi}{4})$

   (b) $X^f(\omega) = \cos(4\,\omega)$

   Solution:

   (a)

   $$\boxed{x(n) = \frac{1}{\pi}\, \cos\left(\frac{\pi}{4}\, n\right)}$$

   (b)

   $$X(\omega) = 0.5\, e^{-j4\omega} + 0.5\, e^{j4\omega}$$

   $$\boxed{x(n) = 0.5\, \delta(n+4) + 0.5\, \delta(n-4)}$$

---

8. What is the DFT of the following N-point signals?

   (a) $x(n) = 2\, \sin\left(\frac{2\,\pi}{N}\, n\right)$, $0 \le n \le N-1$

(b) $x(n) = 1$, $0 \leq n \leq N-1$

(c)

$$x(n) = \begin{cases} 1 & n = M \\ 0 & n \neq M \end{cases}$$

Solution:

$$x(n) = 2 \sin\left(\frac{2\pi}{N} n\right) = j\, e^{-j\frac{2\pi}{N}n} - j\, e^{j\frac{2\pi}{N}n}$$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}nk}$$

$$X(k) = j \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}n}\, e^{-j\frac{2\pi}{N}nk} - j \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}n}\, e^{-j\frac{2\pi}{N}nk}$$

$$X(k) = j \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}n(k+1)} - j \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}n(k-1)}$$

If $k_o = 0$, or k_o is an integer multiple of N then

$$\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}nk_o} = \sum_{n=0}^{N-1} 1 = N.$$

If $1 \leq k_o \leq N-1$, then (using the geometric sum formula)

$$\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}nk_o} = 0.$$

Therefore

$$X(k) = \begin{cases} -j\,N & k = 1 \\ j\,N & k = N-1 \\ 0 & k = 0; 2 \leq k \leq N-2 \end{cases}$$

$$X(k) = -j\,N\,\delta(k-1) + j\,N\,\delta(k-(N-1))$$

(b)

$$X(k) = \begin{cases} N & k = 0 \\ 0 & 1 \leq k \leq N-1 \end{cases}$$

(c)

$$X(k) = e^{-j\frac{2\pi}{N}\,M\,k}$$

---

9. The following 5 samples of the 9-point DFT $X(k)$ of a real length-9 sequence are given:

$$X(0) = 1.9, \quad X(1) = 2.1 + 7.1j, \quad X(4) = -1.1 - 2.4j,$$

$$X(6) = 0.4 + 0.9j, \quad X(7) = 0.2 - 4j$$

Determine the remaining 4 samples using the appropriate DFT property. What DFT property is used here?

Solution:

If $x(n)$ is real, then $X(k) = \overline{X(-k)}$. We can also use the periodicity of the DFT.

$$X(2) = \overline{X(-2)} = \overline{X(9-2)} = \overline{X(7)} = 0.2 + 0.4j$$
$$X(3) = \overline{X(-3)} = \overline{X(9-3)} = \overline{X(6)} = 0.4 - 0.9j$$
$$X(5) = \overline{X(-5)} = \overline{X(9-5)} = \overline{X(4)} = -1.1 + 2.4j$$
$$X(8) = \overline{X(-8)} = \overline{X(9-8)} = \overline{X(1)} = 2.1 - 7.1j$$

---

10. **Periodic convolution with the DFT**

Let $x(n)$, $g(n)$ be the 2 finite length sequences

$$x(n) = (2, 4, 3, 7) \qquad g(n) = (4, 8, 2, 9)$$

for $n = 0, 1, 2, 3$.

(a) By hand, find the circular convolution of $x(n)$ and $g(n)$ and call it $v(n)$.

(b) Now using Matlab find the DFT of $x(n)$ and $g(n)$, call them $X(k)$ and $G(k)$. Verify the DFT cyclic convolution theorem by showing that the IDFT of $X(k)\,G(k)$ is $v(n)$. Use the `fft` command in Matlab to compute the DFT.

Solution:

(a)

$$(2, 4, 3, 7) \circledast (4, 8, 2, 9) =$$
$$2 \cdot (4, 8, 2, 9) + 4 \cdot (9, 4, 8, 2) + 3 \cdot (2, 9, 4, 8) + 7 \cdot (8, 2, 9, 4) =$$
$$(8, 16, 4, 18) + (36, 16, 32, 8) + (6, 27, 12, 24) + (56, 14, 63, 28) =$$
$$(106, 73, 111, 78)$$

(b)
```
>> x = [2 4 3 7];
>> g = [4 8 2 9];
>> X = fft(x);
>> G = fft(g);
>> ifft(X.*G)

ans =

   1.0e+02 *

   1.0600      0.7300-0.0000i      1.1100      0.7800 + 0.0000i

>> round(ans)

ans =

   106    73   111    78
```

11. **Liner convolution with the DFT**

Let $x(n)$, $g(n)$ be the 2 finite length sequences

$$x(n) = (2, 4, 3, 7) \qquad g(n) = (4, 8, 2, 9)$$

for $n = 0, 1, 2, 3$. Compute the linear (usual) convolution

$$y(n) = x(n) * g(n)$$

using the DFT combined with zero padding. Append 3 zeros to each of the two finite-length sequences $x(n)$ and $g(n)$.

$$x_2(n) = (2, 4, 3, 7, 0, 0, 0) \qquad g_2(n) = (4, 8, 2, 9, 0, 0, 0)$$

for $n = 0, 1, 2, \ldots, 6$. Use the MATLAB command `fft` to compute the DFT coefficients $X_2(k)$ and $G_2(k)$, and then take the inverse DFT of $X_2(k)G_2(k)$ using the command `ifft`. Confirm that you obtain the correct linear convolution of $x(n)$ and $g(n)$.

Solution:

```
>> x2 = [2 4 3 7 0 0 0];
>> g2 = [4 8 2 9 0 0 0];
>> X2 = fft(x2);
>> G2 = fft(g2);
>> ifft(X2.*G2)

ans =

  Columns 1 through 4

   8.0000 - 0.0000i  32.0000 - 0.0000i  48.0000 + 0.0000i  78.0000

  Columns 5 through 7

  98.0000 + 0.0000i  41.0000 - 0.0000i  63.0000 - 0.0000i

>> round(ans)

ans =

     8    32    48    78    98    41    63

>> conv([2 4 3 7],[4 8 2 9])

ans =

     8    32    48    78    98    41    63

>> % They are the same.
```